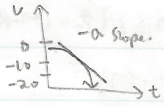
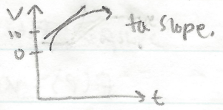
Models

population $\frac{dP}{dt} = kP$ K = births − deaths $\Rightarrow P(t) = Ce^{kt}$

Acceleration $\frac{d^2x}{dt^2} = \frac{dv}{dt} = a(t)$ If a Constant $\Rightarrow x(t) = \frac{a}{2}x^2 + \underbrace{v_0}_{v(t_0)} x + \underbrace{x_0}_{x(t_0)}$

↑ Choose as +ve direction −a if accelerating (downwards), +a if decelerating (upwards). +v if Moving in this direction





Separable eqns $\frac{dy}{dx} = \frac{g(x)}{f(y)} \Leftrightarrow f(y)dy = g(x)dx \Leftrightarrow \int f(y)dy = \int g(x)dx \Rightarrow F(y) = G(x) + C$ ← implicit soln.

growth $\frac{dP}{dt} = kP \Leftrightarrow P(t) = P_0 e^{k(t-t_0)}$ (P_0, t_0) initial conditions.

Cooling / Heating $\frac{dT}{dt} = k(A - T)$ A = temp of environment, T = temp of object

linear 1st order ODE $\frac{dy}{dx} + P(x)y = Q(x) \Rightarrow$ Multiply by $e^{\int P(x)dx}$ the solve. $\Leftrightarrow \frac{d}{dx}\left[e^{\int P(x)dx} y\right] = e^{\int P(x)dx} Q(x)$

Mixture problems $\frac{dx}{dt} = C_i r_i - \frac{X(t)}{V} r_o$ or $\frac{dx}{dt} = C_i r_i - \frac{X(t)}{V + (r_i - r_o)t} r_o$

Existence and Uniqueness given $f(x,y) = \frac{dy}{dx}$, $y(a) = b$: if f continuous near $(a,b) \Rightarrow$ existence of solution $y(x)$.
if $\frac{\partial f}{\partial y}$ continuous near $(a,b) \Rightarrow$ uniqueness of solution $y(x)$.

Substitution $\frac{dy}{dx} = f(ax + by + c)$; $v = ax + by + c$ $y = \frac{v - ax - c}{b}$ $\frac{dv}{dx} = a + b\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{b}\frac{dv}{dx} - \frac{a}{b}$

$\Rightarrow \frac{1}{b}\frac{dv}{dx} - \frac{a}{b} = f(v) \Rightarrow \frac{dv}{dx} = a + bf(v) \Rightarrow \int \frac{dv}{a + bf(v)} = \int dx$

homogeneous $\frac{dy}{dx} = F(y/x)$ $v = y/x$, $y = vx$ $\frac{d}{dx}(vx) = \frac{dv}{dx}x + v$, $\frac{dv}{dx}x = F(v) - v \Rightarrow \int \frac{dv}{F(v) - v} = \int \frac{1}{x}dx$

Exact Solution $\frac{dF(x,y)}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \Leftrightarrow \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$ } differential form.

$M(x,y)dx + N(x,y)dy = 0$; exact iff $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$ for all points of R

integrate $M(x,y)$ w.r.t. x to get $g(y)$, then differentiate w.r.t y to get $g(y')$. Then equate to $N(x,y)$ to figure out $g(y')$, then $\int y'dy$ to get $g(y)$.

logistic equation $\frac{dP}{dt} = aP - bP^2 = kP(M-P)$ $\quad k=b,\ M=\frac{a}{b}\quad P_0=P(0)$

$$P(t) = \frac{MP_0}{P_0+(M-P_0)e^{-kMt}}$$

$P_0 < M$ $\quad P(t)$ increases to M

$P_0 > M$ $\quad P(t)$ decreases to M

$\frac{dP}{dt} = (\beta-\delta)P = kP(M-P)$ $\quad \beta =$ constant, $\delta = \alpha P$ [Competition]

$= (\beta-\delta)P = k(M-P)$ [limited environment]

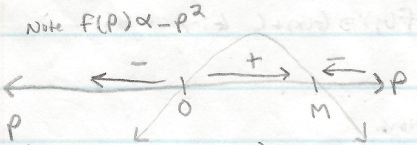
$= kP(M-P)$ [Joint proportion] $\quad P(t) \to$ has disease; $M-P(t) \to$ doesn't have disease

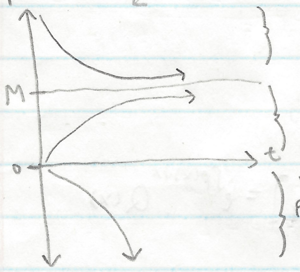
for $P(t) = MP_0/(P_0+(M-P_0)e^{-kMt})$

$f(P) = kP(M-P)$

* $-$ve $\Rightarrow$ go left
* $+$ve $\Rightarrow$ go right

Note $f(P) \propto -P^2$





$\lim_{t\to\infty} \frac{MP_0}{P_0+[-ve\,\#]e^{-kMt}} = M$

$\lim_{t\to\infty} \frac{MP_0}{P_0+[+ve\,\#]e^{-kMt}} = M$

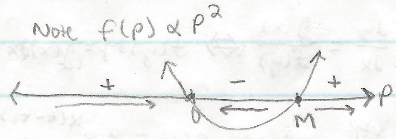
let $P_0=-1$ $\quad \lim_{t\to\infty} \frac{-M}{(M-1)e^{-kMt}-1} = -\infty$

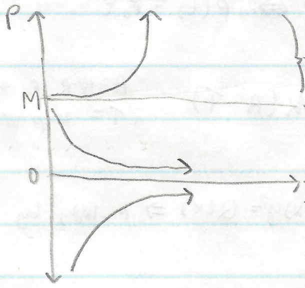
Since $\lim_{t\to t^*} (M-1)e^{-kMt}-1=0$

for $P(t) = MP_0/(P_0+(M-P_0)e^{kMt})$

$f(P) = kP(P-M)$

Note $f(P) \propto P^2$





$\lim_{t\to\infty} \frac{MP_0}{P_0+[-ve\,\#]e^{kMt}} = \infty$

$\lim_{t\to\infty} \frac{MP_0}{P_0+[+ve\,\#]e^{kMt}} = 0$

let $P_0=-1$ $\quad \lim_{t\to\infty} \frac{-M}{(M-1)e^{kMt}-1} = 0$

Harvesting $\frac{dx}{dt} = kx(M-x)-h$

critical points: $kx(M-x)-h=0 \Rightarrow -kx^2+kMx-h=0 \Rightarrow \overbrace{H,N}^{\text{solutions}} = \frac{kM\pm\sqrt{(kM)^2-4kh}}{2k} = \frac{1}{2}\left(M\pm\sqrt{M^2-4h/k}\right)$

$\Rightarrow \frac{dx}{dt} = k(N-x)(x-H)$ w/ $0<H<N<M$

limiting solution (N)
threshold solution (H)

linear 2nd order i) $A(x)y'' + B(x)y' + C(x)y = D(x)$ ← nonhomogeneous. [ii) is homogeneous equation associated with i)]

ii) $A(x)y'' + B(x)y' + C(x)y = 0$ ← homogeneous.

Superimposition for Homogeneous sol'ns

$\underbrace{y}_{\text{new sol'n}} = c_1\underbrace{y_1}_{\text{sol'n 1}} + c_2\underbrace{y_2}_{\text{sol'n 2}}$

Existence and Uniqueness for linear Eqns

If p, q, f continuous on open interval I containing a, then given any 2 numbers b_0 and b_1, $y'' + p(x)y' + q(x)y = f(x)$ has a unique sol'n on entire interval I w/ $y(a)=b_0$, $y'(a)=b_1$

General Solution of Homogeneous equations

A general solution $y = c_1y_1 + c_2y_2$ for y_1, y_2 linearly independent solutions.

*Can have multiple diff. general solutions.

linear independence

f, g linearly indp functions $\Leftrightarrow \frac{f}{g} = \text{constant}$ or $\frac{g}{f} = \text{constant}$ on I.

Wronksian (2x2)

$$W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$$

if f, g 2 soln's of homogeneous 2nd order linear equation on open interval I with f, g continuous

if f, g linearly dependent, $W(f, g) = 0$

if f, g linearly independent soln's to $W(f, g) \neq 0$ at each point on I.

linear 2nd order w/ Constants

$ay'' + by' + cy = 0 \Rightarrow C(r) = ar^2 + br + c = 0 \Leftrightarrow$ Characteristic eqn w/ roots r_1, r_2

$\Rightarrow$ if r_1, r_2 distinct, real $\Rightarrow y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is a general solution

$\Rightarrow$ if $r_1 = r_2$ real $\Rightarrow y(x) = (c_1 + c_2 x) e^{r_1 x}$

$\Rightarrow$ if r_1, r_2 complex $r_1 = a + bi$; $r_2 = a - bi \Rightarrow y(x) = e^{ax} \cos bx \, c_1 + e^{ax} \sin bx \, c_2$

Wronksian (nxn)

$$\left| \begin{bmatrix} f_1 & f_2 & & f_n \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & & f_n^{(n-1)} \end{bmatrix} \right| = W(f_1, f_2, \ldots, f_n) = 0 \text{ if } \{f_1, \ldots, f_n\} \text{ linearly dependent.}$$

$y(x) = y_c + y_p$

Eulers Method

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + h \cdot \overset{y'}{f}(x_n, y_n)$$

(?) $a \pm bi$ = Spiral ($a<0$ = unstable)
bi = centre

Nonlinear-Systems

$\lambda_1 < \lambda_2 < 0$ Stable improper node.

$\lambda_1 = \lambda_2 < 0$ Stable node or Spiral point. — defective $\lambda \Rightarrow$ improper — $\dim(A-\lambda I)=2 \Rightarrow$ proper

$\lambda_1 < 0 < \lambda_2$ unstable saddle point.

$\lambda_1 = \lambda_2 > 0$ unstable node or spiral point. — defective $\lambda \Rightarrow$ improper — $\dim(A-\lambda I)=2 \Rightarrow$ proper

$\lambda_1 > \lambda_2 > 0$ unstable improper node.

$\lambda_1, \lambda_2 = a \pm bi$ Spiral point ($a<0$ = stable, $a>0$ = unstable).

$\lambda_1, \lambda_2 = \pm bi$ Stable or unstable, Centre or Spiral point.

unstable = Source, Stable = Sink

$Ax = x'$ 2D system w/ c.p. (0,0) — λ_1, λ_2 = pure imaginary = Centre.

Linear Systems — $\lambda_1, \lambda_2 < 0$ (real parts) — Asymptotically Stable
— $\lambda_1, \lambda_2 = qi$ — Stable, not Asymptotically Stable
— $\lambda_1 > 0$ or $\lambda_2 > 0$ (real parts) — unstable
* Asymptotically stable $\Rightarrow$ stable

$e^{At} = [x_1(t) \dots x_n(t)]([x_1(0) \dots x_n(0)])^{-1}$, where $[x_1(t) \dots x_n(t)]$ sol'n to $x' = Ax$

$e^{At} = I + At + \dots + A^n \frac{t^n}{n!}$ if A nilpotent and $A^{n+1} = 0$

Sol'n to $x' = Ax$ with initial value $x(0) = \underline{x_0}$ is $x(t) = e^{At}\underline{x_0}$

given $f(x,y) = \frac{dy}{dx}$ $y(a) = b$

f cont near $(a,b) \Rightarrow$ a sol'n exists

$\frac{\partial f}{\partial y}$ cont near $(a,b) \Rightarrow$ sol'n is unique

$\frac{dy}{dt} = f(x,y)$, $\frac{dx}{dt} = g(x,y)$

$$\gamma = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$$